

FROM GAUSSIAN ESTIMATES FOR NONLINEAR EVOLUTION EQUATIONS TO THE LONG TIME BEHAVIOR OF BRANCHING PROCESSES

LUCIAN BEZNEA, LIVIU I. IGNAT, AND JULIO D. ROSSI

ABSTRACT. We study solutions to the evolution equation $u_t = \Delta u - u + \sum_{k \geq 1} q_k u^k$, $t > 0$, in \mathbb{R}^d . Here the coefficients $q_k \geq 0$ verify $\sum_{k \geq 1} q_k = 1 < \sum_{k \geq 1} k q_k < \infty$. First, we deal with existence, uniqueness, and the asymptotic behavior of the solutions as $t \rightarrow +\infty$. We then deduce results on the long time behavior of the associated branching process, with state space the set of all finite configurations of \mathbb{R}^d , under the assumption that $\sum_{k \geq 1} k^2 q_k < \infty$. It turns out that the distribution of the branching process behaves when the time tends to infinity like that of the Brownian motion on the set of all finite configurations of \mathbb{R}^d . However, due to the lack of conservation of the total mass of the initial non linear equation, a deformation with a multiplicative coefficient occurs. Finally, we establish asymptotic properties of the occupation time of this branching process.

Keywords: Branching process, occupation time, long time behavior, space of finite configurations, branching kernel, nonlinear PDE.

2010 MSC: 35J60, 60J45, 60J68, 60J80, 60J35, 47D07.

1. INTRODUCTION

In this paper we consider the following evolution problem

$$(1) \quad \begin{cases} u_t(x, t) = \Delta u(x, t) - u(x, t) + \sum_{k \geq 1} q_k u^k(x, t), & x \in \mathbb{R}^d, t > 0, \\ 0 \leq u(x, t) \leq 1, \\ u(x, 0) = \varphi(x), \end{cases}$$

and analyze the long time behavior of the solutions. We assume that $q_k, k \geq 1$, are nonnegative numbers satisfying

$$(2) \quad \sum_{k \geq 1} q_k = 1,$$

and

$$(3) \quad 1 < \sum_{k \geq 1} k q_k = q < \infty.$$

The main results obtained here regarding equation (1) are stated below.

Theorem 1.1. *For any initial datum $\varphi \in L^\infty(\mathbb{R}^d)$ such that $0 \leq \varphi \leq 1$ there exists a unique global mild solution $u \in C([0, \infty), L^\infty(\mathbb{R}^d))$ to (1) that satisfies $0 \leq u(x, t) \leq 1$ for every $x \in \mathbb{R}^d, t \geq 0$, and*

$$(4) \quad \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \leq \|\varphi\|_{L^\infty(\mathbb{R}^d)}.$$

If in addition $\varphi \in L^1(\mathbb{R}^d)$ then $u \in C([0, \infty), L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$, and

$$(5) \quad \int_{\mathbb{R}^d} u(x, t) dx \leq \int_{\mathbb{R}^d} \varphi(x) dx.$$

Moreover, for any $\varphi_1, \varphi_2 \in L^1(\mathbb{R}^d)$ with $0 \leq \varphi_1, \varphi_2 \leq 1$, the corresponding solutions u_1 and u_2 satisfy

$$(6) \quad \|u_1 - u_2\|_{L^\infty([0, T], L^1(\mathbb{R}^d))} \leq C(T, q) \|\varphi_1 - \varphi_2\|_{L^1(\mathbb{R}^d)}.$$

We emphasize that for initial data $\varphi \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, classical regularity results for the linear heat equation and a bootstrap argument guarantee that in fact the mild solution obtained above satisfies $u \in C((0, \infty), W^{2,p}(\mathbb{R}^d)) \cap C^1((0, \infty), L^p(\mathbb{R}^d))$ for all $1 \leq p \leq \infty$.

Once the well-posedness of system (1) is established, we analyze the asymptotic behavior of the solutions. Assumption (3) implies that $q_1 < 1$, otherwise the system is trivially reduced to a linear problem. We use p for the index of the first nonzero term, q_p , of $(q_k)_{k \geq 2}$.

Theorem 1.2. *For any $\varphi \in L^1(\mathbb{R}^d)$ with $0 \leq \varphi \leq 1$ the solution to (1) satisfies*

$$(7) \quad e^{-(1-q_1)t}(K_t * \varphi)(x) \leq u(x, t) \leq C(d, p, \|\varphi\|_{L^1(\mathbb{R}^d)})e^{-(1-q_1)t}(K_t * \varphi)(x), \quad x \in \mathbb{R}^d,$$

where K_t is the heat kernel. Moreover, there exists a positive constant C_φ , the total asymptotic mass, such that

$$(8) \quad t^{d/2} \|e^{t(1-q_1)}u(\cdot, t) - C_\varphi K_t\|_{L^\infty(\mathbb{R}^d)} \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Now we move to a probabilistic interpretation of our results. Recall that equation (1) is related to the discrete branching Markov processes with state space the set $\widehat{\mathbb{R}^d}$ of all positive measures on \mathbb{R}^d which are finite sums of Dirac measures, called the *space of finite configurations* of \mathbb{R}^d , with the following probabilistic interpretation. An initial particle starts at a point of \mathbb{R}^d and moves according to the d -dimensional Brownian motion, until a random time when it is destroyed and replaced by a finite number of new particles, its direct descendants. The number q_k is the probability that a particle destroyed has precisely k descendants. Each direct descendant starts at the terminal position of the parent particle and moves again according to the d -dimensional Brownian motion until its own terminal time when it is also destroyed and replaced by a new generation of particles and the process continues in this manner.

Recall that the d -dimensional Brownian motion induces in a canonical way a diffusion Markov process with state space the set of finite configurations, describing the movement of the systems of particles from $\widehat{\mathbb{R}^d}$ without having any branching: a system of k particles moves according to k independent Brownian motions; we call it *Brownian motion on $\widehat{\mathbb{R}^d}$* . The probabilistic interpretation of our results is the following: the long time asymptotic behavior of the distribution of the branching process on the space of finite configurations of \mathbb{R}^d , associated with the sequence $(q_k)_{k \geq 1}$ and having as base process the d -dimensional Brownian motion, is the distribution of the $(1 - q_1)$ -subprocess of the Brownian motion on $\widehat{\mathbb{R}^d}$, multiplied with the corresponding asymptotic mass. As far as the authors know, in the case of equation (1) there is no explicit dependence of the asymptotic mass C_φ in terms of φ . In the case of the linear heat equation, i.e. $q_1 = 1$, the dependence is explicit $C_\varphi = \int_{\mathbb{R}^d} \varphi$.

Finally, we consider the issue of occupation times, that is, the times spent by the process in given subsets of the state space during a finite interval of time. Let us denote by \widehat{X} the branching processes induced by the solution $u(x, t)$ obtained in Theorem 1.1. We consider

weighted occupation times with respect functions $\varphi \in b\mathcal{B}(\mathbb{R}^d)$. Following [12], we define the *weighted occupation time process* $Y_t(\varphi) : \Omega \rightarrow \mathbb{R}$ as

$$Y_t(\varphi) := \int_0^t \langle \varphi, \widehat{X}_s \rangle ds, \quad t \geq 0,$$

where Ω is the path space of \widehat{X} . This process is a.s. real-valued.

Here, we provide some estimates on this occupation time process. In fact, assuming $\sum_{k \geq 1} k^2 q_k < \infty$, for any $\varphi \in C_b(\mathbb{R}^d)$, $\varphi \geq 0$ the weighted occupation time process Y_t satisfies

$$\lim_{t \rightarrow \infty} \mathbb{E}^\mu[\exp(-Y_t(\varphi))] = 0,$$

and for $\alpha < (\sum_{k \geq 1} (k-1)q_k)^{-1}$

$$\lim_{T \rightarrow \infty} \mathbb{E}^\mu \left[\exp\left(-\frac{Y_T(\varphi)}{\exp(T/\alpha)}\right) \right] = 1.$$

This asymptotic behavior of the occupation time of the branching process is based on an integral representation of it, a version of a result from [12] for measure-valued superprocesses.

The paper is organized as follows: in Section 2 we deal with the PDE and prove Theorem 1.1 and Theorem 1.2; in Section 3 we introduce the branching processes on the finite configurations of \mathbb{R}^d and analyze the long-time behavior. Related references involving branching processes and the associated nonlinear PDEs, are given in Remark 3.1. Section 4 analyzes the weighted occupation time process induced by the branching process presented in Section 3. Finally, we include an Appendix where we collect basic facts on the right Markov processes and some technical results used in Section 4.

2. WELL-POSEDNESS AND ASYMPTOTIC BEHAVIOR FOR (1)

Let us start showing the well-posedness of our problem.

Proof of Theorem 1.1. Existence and uniqueness of global solutions is not difficult to prove. We include some details here for the sake of completeness. We consider the subset X_T of $C([0, T], L^\infty(\mathbb{R}^d))$ given by

$$X_T = \{C([0, T], L^\infty(\mathbb{R}^d)), 0 \leq u \leq 1\}$$

endowed with the following norm

$$\|u\|_{X_T} = \max_{t \in [0, T]} \|u(t)\|_{L^\infty(\mathbb{R}^d)}.$$

We consider the semigroup $S(t)$ associated with the following linear problem

$$(9) \quad \begin{cases} u_t(x, t) = \Delta u(x, t) - u(x, t), & x \in \mathbb{R}^d, t > 0, \\ u(x, 0) = \varphi(x). \end{cases}$$

It follows that

$$S(t)\varphi = e^{-t}(K_t * \varphi)$$

where K_t is the heat kernel. Observe that for any $p \in [1, \infty]$ we have

$$(10) \quad \|S(t)\varphi\|_{L^p(\mathbb{R}^d)} = e^{-t}\|K_t * \varphi\|_{L^p(\mathbb{R}^d)} \leq e^{-t}\|\varphi\|_{L^p(\mathbb{R}^d)}.$$

Define $\Phi_u(t)$ as follows:

$$\Phi_u(t) = S(t)\varphi + \int_0^t S(t-s) \left(\sum_{k \geq 1} q_k u^k(s) \right) ds.$$

It is easy to see that if $0 \leq \varphi \leq 1$ and $u \in X_T$ then $\Phi_u(t) \geq 0$. Also,

$$\begin{aligned} \|\Phi_u(t)\|_{L^\infty(\mathbb{R}^d)} &\leq e^{-t} \|\varphi\|_{L^\infty(\mathbb{R}^d)} + \int_0^t e^{-(t-s)} \left\| \sum_{k \geq 1} q_k u^k(s) \right\|_{L^\infty(\mathbb{R}^d)} ds \\ &\leq e^{-t} + \sum_{k \geq 1} q_k \int_0^t e^{-(t-s)} ds = 1. \end{aligned}$$

It follows that $\Phi_u : X_T \rightarrow X_T$ is well defined. Let us now choose $u, v \in X_T$. We have that

$$\begin{aligned} \|\Phi_u(t) - \Phi_v(t)\|_{L^\infty(\mathbb{R}^d)} &\leq \int_0^t e^{-(t-s)} \sum_{k \geq 1} q_k \|u^k(s) - v^k(s)\|_{L^\infty(\mathbb{R}^d)} ds \\ &\leq \sum_{k \geq 1} k q_k \int_0^t e^{-(t-s)} \|u(s) - v(s)\|_{L^\infty(\mathbb{R}^d)} ds \leq (1 - e^{-t}) q \|u(s) - v(s)\|_{X_T}. \end{aligned}$$

Hence, it follows that there exists a time $T_0 = T_0(q)$ such that Φ_u is a contraction on X_{T_0} . Then there exists a unique solution u of the fixed point problem $u = \Phi_u$ in the set X_{T_0} . Since T_0 can be fixed independently of the initial data φ and $u(T_0)$ belongs to $L^\infty(\mathbb{R}^d)$ with $0 \leq u(T_0) \leq 1$ we can repeat the same procedure to obtain a global solution.

When φ also belongs to $L^1(\mathbb{R})$ the same argument as above allows to construct a solution in $C([0, T], L^1(\mathbb{R}) \cap X_T)$. Since we have uniqueness in X_T this implies that the new obtained solution is exactly the one constructed previously in X_T .

Let us now prove the other two properties of u given in the theorem. Estimates (5) and (6) shows the global existence of solutions of equation (1) in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

The second one concerning the mass of the solution is immediate since $u \in [0, 1]$ and $\sum_{k \geq 1} q_k = 1$:

$$\frac{d}{dt} \int_{\mathbb{R}^d} u(x, t) dx = \int_{\mathbb{R}^d} \left(-u(x, t) + \sum_{k \geq 1} q_k u^k(x, t) \right) dx \leq 0.$$

In order to prove (4) we consider the case of $L^1(\mathbb{R})$ -solutions since once (4) is obtained for such solutions then by an approximation argument we can extend it to $L^\infty(\mathbb{R})$ -solutions. Let us denote by $M = \|\varphi\|_{L^\infty(\mathbb{R}^d)} \leq 1$. Then

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} (u(t) - M)^+ dx &= \int_{\mathbb{R}^d} u_t \operatorname{sgn}(u(t) - M)^+ dx \\ &= \int_{\mathbb{R}^d} \Delta u (u(t) - M)^+ dx + \int_{\mathbb{R}^d} \left(\sum_{k \geq 1} q_k u^k - u \right) \operatorname{sgn}(u(t) - M)^+ dx \\ &\leq \int_{M \leq u \leq 1} \left(\sum_{k \geq 1} q_k u^k - u \right) dx \leq 0. \end{aligned}$$

Hence for any $t > 0$, $(u(t) - M)^+ = 0$ and we conclude that $u(t) \leq M$.

Let us now prove the contraction property (6). Using property (10) of the semigroup $S(t)$ and the fact that $0 \leq u_1, u_2 \leq 1$ we have

$$\begin{aligned} \|u_1(t) - u_2(t)\|_{L^1(\mathbb{R}^d)} &\leq e^{-t} \|\varphi_1 - \varphi_2\|_{L^1(\mathbb{R}^d)} + \sum_{k \geq 1} \int_0^t e^{-(t-s)} q_k \|u_1^k(s) - u_2^k(s)\|_{L^1(\mathbb{R}^d)} ds \\ &\leq e^{-t} \|\varphi_1 - \varphi_2\|_{L^1(\mathbb{R}^d)} + \sum_{k \geq 1} k q_k \int_0^t e^{-(t-s)} \|u_1(s) - u_2(s)\|_{L^1(\mathbb{R}^d)} ds. \end{aligned}$$

Hence, using Gronwall's lemma we obtain that

$$\|u_1(t) - u_2(t)\|_{L^1(\mathbb{R}^d)} \leq \|\varphi_1 - \varphi_2\|_{L^1(\mathbb{R}^d)} e^{(q-1)t} \text{ for all } t \geq 0,$$

as we wanted to show. \square

We now turn our attention to the asymptotic behavior of the solutions as $t \rightarrow \infty$ and prove Theorem 1.2.

Proof of Theorem 1.2. Step I. Proof of estimate (7). We first observe that the solution u of (1) is a subsolution for the heat equation, that is, since

$$F(u) = -u + \sum_{k \geq 1} q_k u^k \leq u \left(-1 + \sum_{k \geq 1} q_k \right) = 0,$$

we have that u verifies,

$$\begin{cases} u_t(x, t) \leq \Delta u(x, t), & x \in \mathbb{R}^d, t > 0, \\ u(x, 0) = \varphi(x). \end{cases}$$

Also u is a supersolution for the linear equation

$$\begin{cases} u_t(x, t) \geq \Delta u(x, t) - u(x, t) + q_1 u(x, t), & x \in \mathbb{R}^d, t > 0, \\ u(x, 0) = \varphi(x). \end{cases}$$

By a comparison argument, it follows that

$$(11) \quad e^{-(1-q_1)t} (K_t * \varphi)(x) \leq u(x, t) \leq (K_t * \varphi)(x) \text{ for all } x \in \mathbb{R}^d, t > 0.$$

We now improve the right hand side of the above estimate by using that u satisfies

$$(12) \quad u_t \leq \Delta u + u(q_1 - 1) + u^p \sum_{k \geq p} q_k u^{k-p} \leq \Delta u - u(1 - q_1) + u^p.$$

Since the initial data satisfies $\varphi \in L^1(\mathbb{R}^d)$ the right hand side of (11) gives us

$$u(x, t) \leq \min\{1, (4\pi t)^{-d/2} \|\varphi\|_{L^1(\mathbb{R}^d)}\}.$$

Hence there exists a positive constant $C = C(d, \|\varphi\|_{L^1(\mathbb{R}^d)})$ such that

$$(13) \quad u(x, t) \leq \frac{C}{(1+t)^{d/2}} \text{ for all } t > 0 \text{ and } x \in \mathbb{R}^d.$$

In the following C is a constant that depends on d, p and $\|\varphi\|_{L^1(\mathbb{R}^d)}$ but may change from one line to another. Using estimates (12) and (13) it follows that u satisfies

$$u_t \leq \Delta u - u(1 - q_1) + C u(1+t)^{-\frac{d}{2}(p-1)}.$$

This implies that

$$v(t) = e^{(1-q_1)t}u(t)$$

satisfies $v_t \leq \Delta v + va(t)$ where

$$a(t) = C(1+t)^{-\frac{d}{2}(p-1)}.$$

Denoting $H(t) = \exp(-\int_0^t a(s)ds)$ it follows that Hv is a sub-solution for the classical heat equation and then v satisfies

$$v(t) \leq (K_t * \varphi) \exp\left(\int_0^t a(s)ds\right).$$

Since $d \geq 1$ and $p \geq 2$ are integers it follows that $d(p-1)/2 \geq 1/2$ hence

$$\int_0^t a(s)ds \leq C \int_0^t (1+s)^{-1/2}ds = C(1+t)^{1/2}.$$

It follows that u satisfies

$$(14) \quad u(x, t) \leq (K_t * \varphi)(x) e^{-(1-q_1)t + C\sqrt{t+1}} \leq C \exp\left(-\frac{(1-q_1)t}{2}\right), \quad \forall t \geq 0.$$

Introducing this estimate in (12) we obtain that u satisfies

$$u_t \leq \Delta u + u(q_1 - 1) + Cu \exp\left(-\frac{(1-q_1)(p-1)t}{2}\right), \quad \forall t \geq 0.$$

Denoting

$$a_1(t) = C \exp\left(-\frac{(1-q_1)(p-1)t}{2}\right)$$

and repeating the above argument we get

$$u(x, t) \leq (K_t * \varphi)(x) e^{-(1-q_1)t} \exp\left(\int_0^t a_1(s)ds\right) \leq C e^{-(1-q_1)t} (K_t * \varphi)(x).$$

The proof of estimate (7) is now finished.

Step II. Proof of estimate (8). Recall that v is given by

$$v(t) = e^{t(1-q_1)}u(t).$$

In the following we prove that

$$(15) \quad \lim_{t \rightarrow \infty} t^{d/2} \|v(t) - C_\varphi K_t\|_{L^\infty(\mathbb{R}^d)} = 0.$$

In view of Step I, the function v satisfies

$$(16) \quad K_t * \varphi \leq v(t) \leq C(d, p, \|\varphi\|_{L^1(\mathbb{R})}) K_t * \varphi.$$

Moreover, it verifies the equation

$$(17) \quad \begin{cases} v_t(x, t) = \Delta v(x, t) + \sum_{k \geq p} q_k e^{-t(1-q_1)(k-1)} v^k(x, t), & x \in \mathbb{R}^d, t > 0, \\ v(x, 0) = \varphi(x). \end{cases}$$

The mass of v satisfies

$$(18) \quad \frac{d}{dt} \int_{\mathbb{R}^d} v(x, t) dx = \int_{\mathbb{R}^d} \sum_{k \geq p} q_k e^{-t(1-q_1)(k-1)} v^k(x, t) dx \geq 0.$$

and in view of (16) is uniformly bounded:

$$\int_{\mathbb{R}^d} v(x, t) dx \leq C(d, p, \|\varphi\|_{L^1(\mathbb{R})}) \|\varphi\|_{L^1(\mathbb{R}^d)}.$$

Hence, there exists a positive constant $C_\varphi \geq \|\varphi\|_{L^1(\mathbb{R}^d)}$, such that

$$\lim_{t \rightarrow \infty} \int_{\mathbb{R}^d} v(t) dt = C_\varphi.$$

Observe that in view of (18) the following also holds:

$$\int_0^\infty \int_{\mathbb{R}^d} \sum_{k \geq p} q_k e^{-t(1-q_1)(k-1)} v^k(s) ds = C_\varphi - \int_{\mathbb{R}^d} \varphi.$$

We now prove (15). Let us fix $\varepsilon > 0$ and then choose t_0 large enough such that

$$C_\varphi - \int_{\mathbb{R}} v(x, t_0) dx = \int_{t_0}^\infty \int_{\mathbb{R}^d} \sum_{k \geq p} q_k e^{-t(1-q_1)(k-1)} v^k(s) ds < \varepsilon.$$

It follows that

$$\begin{aligned} \|v(t) - C_\varphi K_t\|_{L^\infty(\mathbb{R}^d)} &\leq \|v(t) - K_t * v(t_0)\|_{L^\infty(\mathbb{R}^d)} + \left\| K_t * v(t_0) - K_t \int_{\mathbb{R}} v(x, t_0) dx \right\|_{L^\infty(\mathbb{R}^d)} \\ &\quad + \|K_t\|_{L^\infty(\mathbb{R}^d)} \left| C_\varphi - \int_{\mathbb{R}} v(x, t_0) dx \right|. \end{aligned}$$

Using the properties of the heat kernel (see for example Lemma 3 in [10]) we have that

$$t^{d/2} \left\| K_t * v(t_0) - K_t \int_{\mathbb{R}} v(x, t_0) dx \right\|_{L^\infty(\mathbb{R}^d)} \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

It remains to prove that for t large enough the following holds

$$t^{d/2} \|v(t) - K_t * v(t_0)\|_{L^\infty(\mathbb{R}^d)} \leq \varepsilon.$$

Since v is solution of (17) for any $t > 2t_0$, we have

$$\begin{aligned} \|v(t) - K_t * v(t_0)\|_{L^\infty(\mathbb{R}^d)} &\leq \int_{t_0}^t \sum_{k \geq p} q_k e^{-s(1-q_1)(k-1)} \|K_{t-s} * v^k(s)\|_{L^\infty(\mathbb{R}^d)} ds \\ &= \int_{t_0}^{t/2} + \int_{t/2}^t = I_1(t) + I_2(t). \end{aligned}$$

For I_1 we have

$$\begin{aligned} I_1(t) &\lesssim \int_{t_0}^{t/2} \sum_{k \geq p} q_k \frac{1}{(t-s)^{d/2}} e^{-s(1-q_1)(k-1)} \|v^k(s)\|_{L^1(\mathbb{R}^d)} ds \\ &\lesssim t^{-d/2} \int_{t_0}^{t/2} \sum_{k \geq p} q_k e^{-s(1-q_1)(k-1)} \|v(s)\|_{L^k(\mathbb{R}^d)}^k ds. \end{aligned}$$

Estimate (16) shows that for t_0 large enough we have $\|v(s)\|_{L^q(\mathbb{R}^d)} \leq 1$ for any $s \geq t_0$ and $q > 1$. Thus

$$\begin{aligned} I_1(t) &\leq t^{-d/2} \sum_{k \geq p} \int_{t_0}^{t/2} q_k e^{-s(1-q_1)(k-1)} ds \\ &\lesssim t^{-d/2} \sum_{k \geq p} q_k e^{-t_0(1-q_1)(k-1)} \leq t^{-d/2} e^{-t_0(1-q_1)} \\ &\leq \varepsilon t^{-d/2}. \end{aligned}$$

In the case of I_2 we use that $\|v(s)\|_{L^q(\mathbb{R}^d)} \leq 1$ for any $s \geq t_0$. Then

$$I_2(t) \leq \int_{t/2}^t \sum_{k \geq p} q_k e^{-s(1-q_1)(k-1)} ds \lesssim e^{-t(1-q_1)/2}.$$

The proof is now finished. \square

2.1. Properties of the asymptotic mass. In this section we analyze some properties of the asymptotic mass C_φ :

$$C_\varphi = \lim_{t \rightarrow \infty} e^{(1-t)q_1} \int_{\mathbb{R}^d} u(x, t) dx.$$

To do that we need the following comparison principle for the solutions of our problem. We say that $\bar{u} \in C([0, T], L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ is a supersolution for problem (1) if \bar{u} satisfies

$$(19) \quad \begin{cases} \bar{u}_t(x, t) \geq \Delta \bar{u}(x, t) - \bar{u}(x, t) + \sum_{k \geq 1} q_k \bar{u}^k(x, t), & x \in \mathbb{R}^d, t > 0, \\ 0 \leq \bar{u}(x, t) \leq 1, \\ \bar{u}(x, 0) \geq \varphi(x). \end{cases}$$

In a similar way we define the subsolution \underline{u} .

Theorem 2.1. *Any sub/super-solutions \underline{u} and \bar{u} that belongs to $C([0, T], L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ satisfy $\underline{u}(t) \leq \bar{u}(t)$ on $[0, T]$.*

Proof. Consider α a positive number that will be chosen latter. The following holds:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} e^{-\alpha t} (\underline{u}(t) - \bar{u}(t))^+ &= \int_{\mathbb{R}^d} (\underline{u}_t - \bar{u}_t) \operatorname{sgn}(\underline{u} - \bar{u})^+ e^{-\alpha t} - \alpha \int_{\mathbb{R}^d} (\underline{u} - \bar{u})^+ e^{-\alpha t} \\ &= e^{-\alpha t} \int_{\mathbb{R}^d} (\Delta \underline{u} - \Delta \bar{u}) \operatorname{sgn}(\underline{u} - \bar{u})^+ \\ &\quad + e^{-\alpha t} \left[\int_{\mathbb{R}^d} (f(\underline{u}) - f(\bar{u})) \operatorname{sgn}(\underline{u} - \bar{u})^+ - \alpha \int_{\mathbb{R}^d} (\underline{u} - \bar{u})^+ \right] \\ &\leq e^{-\alpha t} \left[\int_{\underline{u} > \bar{u}} (f(\underline{u}) - \alpha \underline{u}) - (f(\bar{u}) - \alpha \bar{u}) \right], \end{aligned}$$

where

$$f(u) = -u + \sum_{k \geq 1} q_k u^k.$$

Since \underline{u} and \overline{u} belongs to the interval $[0, 1]$ we can choose a positive α such that the map $u \rightarrow f(u) - \alpha u$ is decreasing in the interval $[0, 1]$. It follows that the right hand side of the above estimate is negative. Thus $\underline{u}(t) \leq \overline{u}(t)$. The proof is now finished. \square

The first property of the asymptotic mass says that the map $\varphi \in L^1(\mathbb{R}^d) \mapsto C_\varphi$ is contractive and the second one involves the convexity of this map.

Theorem 2.2. *The map $\varphi \in L^1(\mathbb{R}^d) \mapsto C_\varphi$ satisfies the following properties:*

1. *For any $\varphi_1, \varphi_2 \in L^1(\mathbb{R}^d)$ with $0 \leq \varphi_1, \varphi_2 \leq 1$ there exists a positive constant $C = C(d, q, p, \|\varphi_1\|_{L^1(\mathbb{R}^d)}, \|\varphi_2\|_{L^1(\mathbb{R}^d)})$ (the constant remains bounded in the balls of radius R of $L^1(\mathbb{R}^d)$) such that the following holds*

$$(20) \quad |C_{\varphi_1} - C_{\varphi_2}| \leq C \|\varphi_1 - \varphi_2\|_{L^1(\mathbb{R}^d)}.$$

2. *For any $\lambda \in (0, 1)$ and any $\varphi_1, \varphi_2 \in L^1(\mathbb{R}^d)$ with $0 \leq \varphi_1, \varphi_2 \leq 1$ we have*

$$(21) \quad C_{\lambda\varphi_1 + (1-\lambda)\varphi_2} \leq \lambda C_{\varphi_1} + (1-\lambda)C_{\varphi_2}.$$

As a consequence of the above theorem we have that the map is continuous, in particular for any sequence $\varphi_n \in L^1(\mathbb{R})$, $0 \leq \varphi_n \leq 1$ such that $\varphi_n \rightarrow \varphi$ in $L^1(\mathbb{R})$ we have $C_{\varphi_n} \rightarrow C_\varphi$.

Proof. Let u_1 and u_2 be solutions to (1) corresponding to the initial data φ_1 and φ_2 . Set $z = v_1 - v_2 = e^{t(1-q_1)}(u_1 - u_2)$. Then z satisfies

$$(22) \quad \begin{cases} z_t(x, t) = \Delta z(x, t) + \sum_{k \geq 1} q_k e^{-t(1-q_1)(k-1)} (v_1^k - v_2^k)(x, t), & x \in \mathbb{R}^d, t > 0, \\ z(x, 0) = \varphi_1(x) - \varphi_2(x). \end{cases}$$

Property (6) in Theorem 1.1 shows that

$$(23) \quad \|z(t)\|_{L^1(\mathbb{R}^d)} \leq C(t, q) \|\varphi_1 - \varphi_2\|_{L^1(\mathbb{R}^d)}.$$

We need to improve this estimate since for large t the constant obtained in the proof of Theorem 1.1 blows-up when the time goes to infinity. Observe that we have

$$|v_1^k - v_2^k| \leq k|v_1 - v_2| \max\{\|v_1\|_{L^\infty(\mathbb{R})}, \|v_2\|_{L^\infty(\mathbb{R})}, \} \leq \tilde{C}^{k-1} k|v_1 - v_2|.$$

for some constant $\tilde{C} = \tilde{C}(\|\varphi_1\|_{L^1(\mathbb{R}^d)}, \|\varphi_2\|_{L^1(\mathbb{R}^d)}, d, p)$ given by estimate (7) in Theorem 1.2. Thus for any $t \geq t_0$ we have

$$\frac{d}{dt} \int_{\mathbb{R}^d} |z(x, t)| dx \leq |v_1(t) - v_2(t)| \left(\sum_{k \geq p} k q_k \tilde{C}^{k-1} e^{-t(1-q_1)(k-1)} \right),$$

where t_0 is large enough such that the last term in the right hand side is finite. Applying Gronwall's inequality we get

$$\|z(t)\|_{L^1(\mathbb{R}^d)} \leq \|z(t_0)\|_{L^1(\mathbb{R}^d)} \exp(A)$$

where

$$A = \int_{t_0}^{\infty} \sum_{k \geq p} q_k \tilde{C}^{k-1} e^{-t(1-q_1)(k-1)} dt = \sum_{k \geq p} \frac{k q_k}{(k-1)(1-q_1)} (\tilde{C} e^{-t_0(1-q_1)})^{k-1} \leq 1$$

provided $\tilde{C} \leq e^{t_0(1-q_1)}$. Using estimate (23) with $t = t_0$ we finally obtain that for any $t > t_0$ the following holds

$$\|z(t)\|_{L^1(\mathbb{R}^d)} \leq C(t_0, q) \|\varphi_1 - \varphi_2\|_{L^1(\mathbb{R}^d)}.$$

Letting now $t \rightarrow \infty$ we obtain that

$$|C_{\varphi_1} - C_{\varphi_2}| \leq C(d, p, q, \|\varphi_1\|_{L^1(\mathbb{R}^d)}, \|\varphi_2\|_{L^1(\mathbb{R}^d)}) \|\varphi_1 - \varphi_2\|_{L^1(\mathbb{R}^d)}$$

which proves the first part of the theorem.

Let us now consider the convexity property of the map $\varphi \mapsto C_\varphi$. Let us denote by w the solution of

$$(24) \quad \begin{cases} w_t(x, t) = \Delta w(x, t) - w(x, t) + \sum_{k \geq 1} q_k w^k(x, t), & x \in \mathbb{R}^d, t > 0, \\ w(x, 0) = \lambda \varphi_1(x) + (1 - \lambda) \varphi_2(x), \end{cases}$$

and by u_1 and u_2 the solutions of (1) corresponding to initial data φ_1 and φ_2 . Let

$$U = \lambda u_1 + (1 - \lambda) u_2.$$

This new function U satisfies

$$\begin{aligned} U_t &= \Delta U - U + \sum_{k \geq 1} q_k \left(\lambda u_1^k + (1 - \lambda) u_2^k \right) \\ &\geq \Delta U - U + \sum_{k \geq 1} q_k (\lambda u_1 + (1 - \lambda) u_2)^k \\ &= \Delta U - U + \sum_{k \geq 1} q_k U^k. \end{aligned}$$

This show that U is a super-solution for system (24) and by maximum principle given by Theorem 2.1 we obtain $U \geq w$. Hence,

$$\int_{\mathbb{R}} w(x, t) dx \leq \lambda \int_{\mathbb{R}} u_1(x, t) dx + (1 - \lambda) \int_{\mathbb{R}} u_2(x, t) dx.$$

Multiplying the above inequality with $e^{(1-q_1)t}$ and letting $t \rightarrow \infty$ we get

$$C_{\lambda \varphi_1 + (1-\lambda) \varphi_2} \leq \lambda C_{\varphi_1} + (1 - \lambda) C_{\varphi_2}.$$

The proof is now complete. \square

3. BRANCHING PROCESSES ON THE FINITE CONFIGURATIONS OF \mathbb{R}^d

Let $E := \mathbb{R}^d$ and define the set \widehat{E} of finite positive measures on E as

$$\widehat{E} := \left\{ \sum_{1 \leq k \leq k_0} \delta_{x_k} : k_0 \in \mathbb{N}^*, x_k \in E \text{ for all } 1 \leq k \leq k_0 \right\} \cup \{\mathbf{0}\},$$

where $\mathbf{0}$ denotes the zero measure. The set \widehat{E} is identified with the union of all symmetric m -th powers $E^{(m)}$ of E , $\widehat{E} = \bigcup_{m \geq 0} E^{(m)}$, where $E^{(0)} := \{\mathbf{0}\}$, and it is called the *space of finite configurations of E* . \widehat{E} is endowed with the topology of disjoint union of topological spaces and the corresponding Borel σ -algebra is denoted by $\mathcal{B}(\widehat{E})$; see, e.g., [11, 7, 5].

A Markov process \overline{X} with state space \widehat{E} is called *branching process* provided that for all $\mu_1, \mu_2 \in \widehat{E}$, the process $X^{\mu_1 + \mu_2}$ starting from $\mu_1 + \mu_2$ and the sum $X^{\mu_1} + X^{\mu_2}$ are equal in distributions.

Branching kernel on \widehat{E} . The *convolution* of two finite measures p_1, p_2 on \widehat{E} is the finite measure $p_1 * p_2$ on \widehat{E} defined by

$$\int p_1 * p_2(d\nu) \mathbf{f}(\nu) := \int p_1(d\nu_1) \int p_2(d\nu_2) \mathbf{f}(\nu_1 + \nu_2), \quad \mathbf{f} \in bp\mathcal{B}(\widehat{E}).$$

A *branching kernel* is a kernel N on \widehat{E} such that for all $\mu, \nu \in \widehat{E}$ we have $N_{\mu+\nu} = N_\mu * N_\nu$, where N_μ denotes the measure on \widehat{E} such that $N\mathbf{f}(\mu) = \int \mathbf{f} dN_\mu$ for all $\mathbf{f} \in bp\mathcal{B}(\widehat{E})$.

Recall that a Markov process with state space \widehat{E} is a branching process if and only if its transition function is formed from branching kernels.

Example; the diagonal kernel on \widehat{E} . Let N be a kernel on E which is sub-Markovian (i.e. $N1 \leq 1$) and for every $k \geq 1$ consider the kernel N^k on E^k , the k -times product of N , defined as

$$N^k f(x) := \underbrace{\int \dots \int}_{k\text{-times}} f(y_1, \dots, y_k) N(x_1, dy_1) \dots N(x_k, dy_k)$$

for all $f \in p\mathcal{B}(E^k)$ and $x = (x_1, \dots, x_k) \in E^k$. Let now \mathbf{N} be the sub-Markovian kernel on \widehat{E} defined as

$$\mathbf{N}\mathbf{f} := \sum_{k \geq 0} N^{(k)}(\mathbf{f}|_{E^{(k)}}), \quad \mathbf{f} \in bp\mathcal{B}(\widehat{E}),$$

where $N^{(k)}$, $k \geq 1$ denotes the symmetric k -times power of N , that is the projection on $E^{(k)}$ of the kernel N^k on E^k , and $N^{(0)} := \delta_0$; see [7], Section 4.1 for details. The kernel \mathbf{N} is called *diagonal* and one can check that it is a branching kernel on \widehat{E} .

We present now two branching processes on the space of finite configurations of E . First, we indicate canonical constructions of path continuous Markov processes on \widehat{E} , induced by the d -dimensional Brownian motion (cf., e.g., [7]).

Let $B = (B_t)_{t \geq 0}$ be the d -dimensional Brownian motion and $(P_t)_{t \geq 0}$ be its transition function, the d -dimensional Wiener semigroup on \mathbb{R}^d .

For $k \geq 1$, let $B^k = (B_t^k)_{t \geq 0}$ be the kd -dimensional Brownian motion, regarded as a path continuous Markov process with state space $E^k = \mathbb{R}^{kd}$. Recall that B^k is the k times Cartesian power of the d -dimensional Brownian and let $(P_t^k)_{t \geq 0}$ be its transition function, the kd -dimensional Wiener semigroup. Let $\mathcal{U}^k = (U_\alpha^k)_{\alpha > 0}$ be its resolvent of kernels on E^k ,

$$U_\alpha^k f(x) = \mathbb{E}^x \int_0^\infty e^{-\alpha t} f(B_t^k) dt, \quad f \in p\mathcal{B}(E^k), x \in E^k.$$

For each $k \in \mathbb{N}^*$, $t \geq 0$, and $\alpha > 0$, we consider the projections $P_t^{(k)}$ and $U_\alpha^{(k)}$ on $E^{(k)}$ of the kernels P_t^k and U_α^k on E^k (for details see [7], Section 4.1). Namely, by Proposition 4.1 and Proposition 4.2 in [7] we get that the family $(P_t^{(k)})_{t \geq 0}$ is a sub-Markovian semigroup of kernels on $E^{(k)}$, with the induced resolvent of kernels $\mathcal{U}^{(k)} = (U_\alpha^{(k)})_{\alpha > 0}$. The semigroup of kernels $(P_t^{(k)})_{t \geq 0}$ is the transition function of a right (Markov) process $B^{(k)}$ with state space $E^{(k)}$, the symmetric dk times power of $B = (B_t)_{t \geq 0}$.

Let $(\mathbf{P}_t)_{t \geq 0}$ be the family of kernels on \widehat{E} defined as $\mathbf{P}_t \mathbf{g}|_{E^{(k)}} := P_t^{(k)}(\mathbf{g}|_{E^{(k)}})$, $\mathbf{g} \in p\mathcal{B}(\widehat{E})$. It is the transition function of a Borel right process $\widehat{B} = (\widehat{B}_t)_{t \geq 0}$ with state space \widehat{E} , called *Brownian motion on \widehat{E}* . So,

$$\mathbf{P}_t \mathbf{g}(\mu) = \mathbb{E}^\mu \mathbf{g}(\widehat{B}_t), \quad t > 0, \quad \mu \in \widehat{E}.$$

Note that since the transition function $(\mathbf{P}_t)_{t \geq 0}$ is formed by diagonal kernels (in particular, they are branching kernels), we conclude that $\widehat{B} = (\widehat{B}_t)_{t \geq 0}$ is a branching process on \widehat{E} .

If $\varphi \in p\mathcal{B}(E)$, define the *multiplicative function* $\widehat{\varphi} : \widehat{E} \rightarrow \mathbb{R}_+$ as

$$\widehat{\varphi}(\mathbf{x}) = \begin{cases} \prod_{k \geq 1} \varphi(x_k), & \text{if } \mathbf{x} = (x_k)_{k \geq 1} \in \widehat{E}, \quad \text{if } \mathbf{x} \neq \mathbf{0}, \\ 1, & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

The next proposition gives the second example of a branching process on the set of all finite configurations of \mathbb{R}^d . More precisely, the solution $u(x, t)$, $t \geq 0$, $x \in E$, of (1) given by Theorem 1.1, induces a branching processes on \mathbb{R}^d .

Proposition 3.1. *There exists a branching process $\widehat{X} = (\widehat{X}_t)_{t \geq 0}$ with state space $\widehat{\mathbb{R}^d}$ such that if we denote by $(\mathbf{H}_t)_{t \geq 0}$ its transition function, then we have*

$$\mathbf{H}_t \widehat{\varphi} = \widehat{u(\cdot, t)} \text{ for all } 0 \leq \varphi \leq 1.$$

Proof. The assertion follows by Theorem 4.10 from [5]. Note that the hypothesis (*) from that theorem is satisfied in this case, due to Proposition 4.9 from the same paper. \square

We can state now the probabilistic interpretation of Theorem 1.2, in terms of branching processes on the space of all finite configurations of \mathbb{R}^d .

Theorem 3.1. *For any $\varphi \in L^1(\mathbb{R}^d)$, $0 \leq \varphi \leq 1$, there exists a positive constant $C(\varphi)$ such that*

$$t^{d/2} |e^{t(1-q_1)} \mathbf{H}_t \widehat{\varphi} - \widehat{C(\varphi)} \mathbf{P}_t \widehat{\varphi}| \leq l_1 o(1),$$

where $(\mathbf{P}_t)_{t \geq 0}$ is the transition function of the Brownian motion on $\widehat{\mathbb{R}^d}$, induced by the d -dimensional Brownian motion.

Proof. Recall first that if $0 \leq \varphi, \psi \leq 1$ then

$$|\widehat{\varphi}(x) - \widehat{\psi}(x)| \leq l_1 \|\varphi - \psi\|_\infty, \quad \forall x \in \widehat{E},$$

where l_1 is the linear functional defined on \widehat{E} as: $l_1|_{E^{(k)}} := k$, $k \geq 0$. Taking into account that by Proposition 3.1 $\mathbf{H}_t \widehat{\varphi} = \widehat{u(\cdot, t)}$ and since we also have, $\mathbf{P}_t \widehat{\varphi} = \widehat{P_t \varphi}$, we can use Theorem 1.2 to obtain

$$t^{d/2} |e^{t(1-q_1)} \mathbf{H}_t \widehat{\varphi} - \widehat{C(\varphi)} \mathbf{P}_t \widehat{\varphi}| \leq l_1 t^{d/2} \|e^{t(1-q_1)} u(\cdot, t) - C(\varphi) P_t \varphi\|_{L^\infty(\mathbb{R}^d)} \leq l_1 o(1).$$

\square

Corollary 3.1. *Let $A \in \mathcal{B}(\mathbb{R}^d)$, $\mu \in \widehat{\mathbb{R}^d}$, and let $\widehat{X} = (\widehat{X}_t)_{t \geq 0}$ be the branching process with state space $\widehat{\mathbb{R}^d}$, given by Proposition 3.1. Then*

$$|\mathbb{E}^\mu(\widehat{X}_t \in \widehat{A}) - \widehat{C(1_A)} \mathbb{E}^\mu(\widehat{B}_t^{1-q_1} \in \widehat{A})| \leq \frac{l_1 o(1)}{e^{t(1-q_1)} t^{d/2}}, \quad t > 0,$$

where \widehat{B}^{1-q_1} is the $(1 - q_1)$ -subprocess of the Brownian motion \widehat{B} on $\widehat{\mathbb{R}^d}$.

Remark 3.1. (i) Corollary 3.1 gives the claimed probabilistic interpretation of the asymptotic result from Theorem 1.2. In fact, the long time asymptotic behavior of the distribution of the branching process on the space of finite configurations of \mathbb{R}^d , associated with the sequence $(q_k)_{k \geq 1}$ and having as base process the d -dimensional Brownian motion, is the distribution of $(1 - q_1)$ -subprocess of the diagonal process, induced by the d -dimensional Brownian motion, multiplied with the corresponding asymptotic mass. Note that for the considered nonlinear equation the asymptotic mass is not just the mass of the initial datum as it happens in the linear case.

(ii) Recall that in the Introduction we stated the role of the sequence $(q_k)_{k \geq 1}$ in the probabilistic interpretation of the branching mechanism of the process $\hat{X} = (\hat{X}_t)_{t \geq 0}$ from Proposition 3.1. It is possible to consider different, more general, branching mechanisms; see, e.g., [5] and [6]. In particular, the statement of Theorem 1.1 and its proof should be compared with Proposition 4.1 from [5].

(iii) Branching processes mentioned in the above assertion (i) were constructed in [3] and [4], related to the fragmentation equation and respectively to a probabilistic model of the fragmentation phase of an avalanche.

(iv) A nonlinear Dirichlet problem associated to the equation (1) is considered and solved in [7] and [8], while a probabilistic numerical approach is developed in [14].

4. OCCUPATION TIME

In Theorem 1.1 we obtained that for any $\varphi \in b\mathcal{B}(\mathbb{R}^d)$, $0 \leq \varphi \leq 1$, there exists a unique solution that in the following we denote by $(U_t \varphi)_{t \geq 0}$, of equation (1) that satisfies and

$$0 \leq U_t(\varphi) \leq 1.$$

Moreover by Proposition 3.1 it introduces a branching process \hat{X} on $\widehat{\mathbb{R}^d}$. For $f \in bp\mathcal{B}(\mathbb{R}^d)$ we define the function $\tilde{U}_t f \in p\mathcal{B}(\mathbb{R}^d)$ as

$$\tilde{U}_t f := -\ln U_t(e^{-f}).$$

Then by Corollary 4.3, Theorem 4.10 and Remark 4.4 (iii) from [5] the following assertions hold:

(i) The family $(\tilde{U}_t)_{t \geq 0}$ is a nonlinear semigroup on $bp\mathcal{B}(\mathbb{R}^d)$ and $\tilde{U}_t f$ is the solution of the equation

$$(25) \quad \begin{cases} \tilde{U}_t = \Delta \tilde{U} - |\nabla \tilde{U}|^2 + 1 - \sum_{k \geq 1} q_k e^{(1-k)\tilde{U}}, & t > 0, x \in \mathbb{R}^d, \\ \tilde{U}(0) = f. \end{cases}$$

(ii) For any $s \geq 0, \mu \in \widehat{\mathbb{R}^d}, g \in bp\mathcal{B}(\mathbb{R}^d)$ the following holds

$$(26) \quad \mathbb{E}^\mu \exp(-\langle g, \hat{X}_s \rangle) = \exp(-\langle \tilde{U}_s g, \mu \rangle).$$

Following [12], for every $\varphi \in b\mathcal{B}(\mathbb{R}^d)$, we define the *weighted occupation time process* $Y_t(\varphi) : \Omega \rightarrow \overline{\mathbb{R}}$ as

$$Y_t(\varphi) := \int_0^t \langle \varphi, \hat{X}_s \rangle ds, \quad t \geq 0,$$

where Ω is the path space of \widehat{X} . Observe that it is a.s. a real-valued process. Indeed, using Proposition 4.8 from [5] we have

$$\begin{aligned} \mathbb{E}^\mu |Y_t(\varphi)| &\leq \mathbb{E}^\mu Y_t(|\varphi|) = \int_0^t \mathbb{E}^\mu l_{|\varphi|}(\widehat{X}_s) ds = \int_0^t \mathbf{H}_s l_{|\varphi|}(\mu) ds \\ &\leq \|\varphi\|_\infty \int_0^t \mathbf{H}_s l_1(\mu) ds \leq l_1(\mu) \|\varphi\|_\infty \int_0^t e^{(1-q_1)s} ds < \infty. \end{aligned}$$

Regarding the process $Y_t(\varphi)$ we have the following representation.

Theorem 4.1. *Let us assume that $\sum_{k \geq 1} k^2 q_k < \infty$. For any $\mu \in \widehat{\mathbb{R}}^d$, $t \geq 0$, $f \in bp\mathcal{B}(\mathbb{R}^d)$, and $\varphi \in C_b(\mathbb{R}^d)$, $\varphi \geq 0$ the following holds*

$$\mathbb{E}^\mu [\exp(-\langle f, \widehat{X}_t \rangle - Y_t(\varphi))] = \exp(-\langle \widetilde{V}_t(\varphi, f), \mu \rangle),$$

where $\widetilde{V}_t(\varphi, f)$, $t \geq 0$ is the solution to

$$(27) \quad \begin{cases} v_t = \Delta v - |\nabla v|^2 + 1 - \sum_{k \geq 1} q_k e^{(1-k)v} + \varphi, & t > 0, x \in \mathbb{R}^d, \\ v(0) = f. \end{cases}$$

The above result is an integral representation of the solution $\widetilde{V}_t(\varphi, f)$ with respect to the $\widehat{\mathbb{R}}^d$ -valued branching process $(\widehat{X}_t)_{t \geq 0}$ and its weighted occupation time process. This is a version for (non local) branching processes on the space of finite configurations of a result for $M(\mathbb{R}^d)$ -superprocesses from [12], Theorem 3.1. For the reader convenience we include in Appendix 5.2 a sketch of the proof.

Theorem 4.2. *For any nonnegative and bounded $\varphi \not\equiv 0$, and $T > 0$ the solution v_T of the equation*

$$(28) \quad \begin{cases} v_t = \Delta v - v + \sum_{k \geq 1} q_k v^k - \frac{1}{T} v \varphi, \\ v(\cdot, 0) = 1 \end{cases}$$

satisfies

$$(29) \quad \lim_{t \rightarrow \infty} v_T(t, x) = 0 \text{ for all } x \in \mathbb{R}^d$$

and for any $\alpha < 1/F'(1)$

$$(30) \quad \lim_{T \rightarrow \infty} v_T(\alpha \log T, x) = 1 \text{ for all } x \in \mathbb{R}^d.$$

We now obtain some asymptotic properties of the occupation time.

Theorem 4.3. *Let us assume that $\sum_{k \geq 1} k^2 q_k < \infty$. For any $\varphi \in C_b(\mathbb{R}^d)$, $\varphi \geq 0$ the weighted occupation time process Y_t satisfies*

$$\lim_{t \rightarrow \infty} \mathbb{E}^\mu [\exp(-Y_t(\varphi))] = 0,$$

and for $\alpha < (\sum_{k \geq 1} (k-1)q_k)^{-1}$

$$\lim_{T \rightarrow \infty} \mathbb{E}^\mu \left[\exp\left(-\frac{Y_T(\varphi)}{\exp(T/\alpha)}\right) \right] = 1.$$

Proof. Let us choose F to be a finite set of points $x_k \in \mathbb{R}^d$ and $\mu = \sum_{x_k \in F} \delta_{x_k}$. Since we are interested in the asymptotic properties of $Y_t(\varphi)$ we choose $f \equiv 0$ in Theorem 4.1. In the case of the first limit we consider system (28) with $T = 1$ and denote by v its solution. We have

$$\mathbb{E}^\mu[e^{-Y_t(\varphi)}] = \exp(-\langle \tilde{V}_t(\varphi, 0), \mu \rangle) = \exp\left(-\sum_{x_k \in F} \tilde{V}_t(\varphi, 0)(x_k)\right) = \prod_{x_k \in F} v(t, x_k).$$

In view of property (29) we obtain the desired result.

Let us now prove the second limit. Under the same assumptions as before we have

$$\mathbb{E}^\mu\left[\exp\left(-\frac{Y_T(\varphi)}{T}\right)\right] = \exp\left(-\sum_{x_k \in F} \tilde{V}_T\left(\frac{\varphi}{T}, 0\right)(x_k)\right) = \prod_{x_k \in F} v_T(T, x_k),$$

where v_T is the solution of (28). Using (30) we obtain the second property. \square

Proof of Th.4.2. Step I. Proof of (29). We can assume that $x = 0$ and prove the required limit. We now consider the truncated problem in the ball $B_R = \{x \in \mathbb{R}^d \mid |x| < R\}$:

$$(31) \quad \begin{cases} v_t = \Delta v - v + \sum_{k \geq 1} q_k v^k - \frac{1}{T} v \varphi, & (x, t) \in B_R \times (0, \infty), \\ v(x, t) = 1, & (x, t) \in \partial B_R \times [0, \infty), \\ v(x, 0) = 1, & x \in B_R. \end{cases}$$

and denote its solution by v_R . Using the maximum principle we have

$$0 \leq v(t, x) \leq v_R(t, x) \text{ for all } t > 0, x \in \overline{B}_R.$$

Let us now consider the energy functional associated with (31)

$$E(t) = \int_{B_R} \left(\frac{1}{2} |\nabla v|^2 - \frac{1}{2} v^2 + \sum_k q_k \frac{v^{k+1}}{k+1} - \frac{v^2 \varphi}{2T} \right)$$

It is uniformly bounded $-\infty < E(t) \leq E(0)$ for all $t \geq 0$ and satisfies

$$\frac{dE}{dt}(t) = - \int_{B_R} v_t^2 \leq 0.$$

Thus $v_R(t) \rightarrow v_R$ in $H^1(B_R)$, as $t \rightarrow \infty$, where v_R is the unique stationary solution of problem (31). Moreover v_R are uniformly bounded in $C^1(\overline{B}_R)$ and then $v_R(t) \rightarrow v_R$ uniformly in \overline{B}_R when $t \rightarrow \infty$. This implies that for any $\epsilon > 0$ there exists $T_\epsilon > 0$ such that

$$(32) \quad 0 \leq v(t, 0) \leq v_R(t, 0) + \epsilon, \text{ for all } t > T_\epsilon.$$

The stationary solutions v_R satisfy

$$0 < v_{R+1}(x) \leq v_R(x) \leq 1 \text{ for all } x \in \overline{B}_R.$$

Therefore there exists the following limit

$$\lim_{R \rightarrow \infty} v_R(x) = v^\infty(x), \quad x \in \mathbb{R}^d,$$

and v^∞ solves the problem

$$(33) \quad \begin{cases} \Delta v - v + \sum_{k \geq 1} q_k v^k - \frac{1}{T} v \varphi = 0, & x \in \mathbb{R}^d, \\ 0 \leq v \leq 1. \end{cases}$$

Multiplying by v^∞ the above equation and integrating on \mathbb{R}^d we obtain that v^∞ should be a constant. Since $\varphi \not\equiv 0$ the only constant function that solves (33) is $v^\infty \equiv 0$. This and (32) give us the desired result.

Step II. Proof of (30). Let us consider M such that $0 \leq \varphi \leq M$. Then $v \geq \bar{v}$ where \bar{v} is solution of

$$\begin{cases} v_t = \Delta v - v + \sum_{k \geq 1} q_k v^k - \frac{M}{T} v, & x \in \mathbb{R}^d, t > 0, \\ v(\cdot, 0) = 1. \end{cases}$$

The unique solution of this equation depends only on t and then we arrive to the ODE

$$\begin{cases} v_t = -v + \sum_{k \geq 1} q_k v^k - \frac{M}{T} v, \\ v(0) = 1. \end{cases}$$

Observe that

$$F(v) = -v + \sum_{k \geq 1} q_k v^k \geq F'(1)(v - 1)$$

where

$$F'(1) = \sum_{k \geq 1} k q_k - 1 = \sum_{k \geq 1} (k - 1) q_k > 0.$$

This implies that $\bar{v} \geq g_T(T)$ where g_T is the solution of the following ODE:

$$\begin{cases} g_t = \left(F'(1) - \frac{M}{T}\right)g - F'(1), \\ g(0) = 1. \end{cases}$$

Solving explicitly, it gives us that

$$g_T(t) = \frac{T F'(1)}{T F'(1) - M} - \frac{M}{T F'(1) - M} e^{t F'(1)} e^{M t / T}.$$

Choosing $t = \alpha \log T$ with $\alpha < 1/F'(1)$ we get

$$\lim_{T \rightarrow \infty} g_T(T) = 1.$$

This implies that v_T , the solution of (28), satisfies

$$\lim_{T \rightarrow \infty} v_T(\alpha \log T, x) = 1 \text{ for all } x \in \mathbb{R}^d$$

and the proof is now complete. \square

5. APPENDIX

5.1. Right Markov processes. Let E be a metrizable Lusin topological space and \mathcal{B} the Borel σ -algebra on E .

A *transition function* on E is a family $(p_t)_{t \geq 0}$ of kernels on (E, \mathcal{B}) which are sub-Markovian (i.e., $p_t 1 \leq 1$ for all $t \geq 0$), such that $p_0 f = f$ and $p_s(p_t f) = p_{s+t} f$ for all $s, t \geq 0$ and $f \in p\mathcal{B}$. We assume that for all $f \in p\mathcal{B}$ the function $(t, x) \mapsto p_t f(x)$ is $\mathcal{B}([0, \infty)) \otimes \mathcal{B}$ -measurable. We denote by $\mathcal{U} = (U_\alpha)_{\alpha > 0}$ the family of kernels on (E, \mathcal{B}) given by

$$U_\alpha f = \int_0^\infty e^{-\alpha t} p_t f dt.$$

Consequently, $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is a sub-Markovian resolvent of kernels on (E, \mathcal{B}) and it is called *associated with* $(p_t)_{t \geq 0}$.

If $(p_t)_{t \geq 0}$ is a transition function on E and $q > 0$, then the family of kernels $(e^{-qt}p_t)_{t \geq 0}$ is also a transition function on E and its associate resolvent is $\mathcal{U}_q := (U_{q+\alpha})_{\alpha>0}$. Recall that if L is the generator of $(p_t)_{t \geq 0}$, then the generator of $(e^{-qt}p_t)_{t \geq 0}$ is $L - q$.

A *right process with state space* E (associated with the transition function $(p_t)_{t \geq 0}$) is a collection $X = (\Omega, \mathcal{G}, \mathcal{G}_t, X_t, \theta_t, \mathbb{P}^x)$ where: (Ω, \mathcal{G}) is a measurable space, $(\mathcal{G}_t)_{t \geq 0}$ is a family of sub σ -algebras of \mathcal{G} such that $\mathcal{G}_s \subseteq \mathcal{G}_t$ if $s < t$; for all $t \geq 0$, $X_t : \Omega \rightarrow E_\Delta$ is a $\mathcal{G}_t/\mathcal{B}_\Delta$ -measurable map such that $X_t(\omega) = \Delta$ for all $t > t_0$ if $X_{t_0}(\omega) = \Delta$, where Δ is a cemetery state adjoined to E as an isolated point of $E_\Delta := E \cup \{\Delta\}$ and \mathcal{B}_Δ is the Borel σ -algebra on E_Δ ; we set $\zeta(\omega) := \inf\{t \mid X_t(\omega) = \Delta\}$; for each $t \geq 0$, the map $\theta_t : \Omega \rightarrow \Omega$ is such that $X_s \circ \theta_t = X_{s+t}$ for all $s > 0$; for all $x \in E_\Delta$, \mathbb{P}^x is a probability measure on (Ω, \mathcal{G}) such that $x \mapsto \mathbb{P}^x(F)$ is universally \mathcal{B} -measurable for all $F \in \mathcal{G}$; $E^x(f \circ X_0) = f(x)$ and the following *Markov property* holds:

$$\mathbb{E}^x(f \circ X_{s+t} \cdot G) = \mathbb{E}^x(p_t^\Delta f \circ X_s \cdot G)$$

for all $x \in E_\Delta$, $s, t \geq 0$, $f \in p\mathcal{B}_\Delta$ and $G \in p\mathcal{G}_s$, where p_t^Δ is the Markovian kernel on $(E_\Delta, \mathcal{B}_\Delta)$ such that $p_t^\Delta 1 = 1$ and $p_t^\Delta|_E = p_t$; for all $\omega \in \Omega$ the function $t \mapsto X_t(\omega)$ is right continuous on $[0, \infty)$; the filtration $(\mathcal{G}_t)_{t \geq 0}$ is *right continuous* (i.e. $\mathcal{G}_t = \mathcal{G}_{t+} := \bigcap_{s>t} \mathcal{G}_s$) and *augmented* (i.e.

$\mathcal{G}_t = \tilde{\mathcal{G}}_t := \bigcap_\mu \mathcal{G}_t^\mu$, where for every probability measure μ on (E, \mathcal{B}) , \mathcal{G}^μ is the completion of \mathcal{G} with respect to the probability measure $\mathbb{P}^\mu := \int \mathbb{P}^x \mu(dx)$ on (Ω, \mathcal{G}) and \mathcal{G}_t^μ is the completion of \mathcal{G}_t in \mathcal{G}^μ with respect to \mathbb{P}^μ); we assume that for all $\alpha > 0$, every function u which is α -excessive with respect to the resolvent associated with $(p_t)_{t \geq 0}$ and each probability measure μ on (E, \mathcal{B}) , the function $t \mapsto u \circ X_t$ is right continuous on $[0, \infty)$ \mathbb{P}^μ -a.s.

We consider the *natural filtration* associated with X : $\mathcal{F} := \tilde{\mathcal{F}}^0$, $\mathcal{F}_t := \tilde{\mathcal{F}}_t^0$, where $\mathcal{F}^0 := \sigma(X_s \mid s < \infty)$, $\mathcal{F}_t^0 := \sigma(X_s \mid s \leq t)$. It is known that always a right process may be considered with respect to its natural filtration:

$$X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}^x).$$

The sub-Markovian resolvent $\mathcal{U} = (U_\alpha)_{\alpha>0}$ associated with $(p_t)_{t \geq 0}$ is called the *resolvent of the process* X and for all $f \in p\mathcal{B}$, $\alpha > 0$ and $x \in E$ we have

$$U_\alpha f(x) = \mathbb{E}^x \int_0^\zeta e^{-\alpha t} f(X_t) dt,$$

with the convention $f(\Delta) = 0$.

If $q > 0$ then there exists a right process X^q with state space E , associated with the transition function $(e^{-qt}p_t)_{t \geq 0}$, called the *q-subprocess* of X . The resolvent family of the process X^q is $(U_{q+\alpha})_{\alpha>0}$. If \mathcal{L} is the generator of X (e. g., as the generator of the C_0 -resolvent of contractions induced by $(U_\alpha)_{\alpha>0}$ on $L^p(E, m)$, where m is a $(p_t)_{t \geq 0}$ -subinvariant measure), then the generator of the q -subprocess X^q is $\mathcal{L} - q$.

A *stopping time* is a map $T : \Omega \rightarrow \bar{\mathbb{R}}_+$ such that the set $[T \leq t]$ belongs to \mathcal{F}_t for all $t \geq 0$.

Let μ be a σ -finite measure on (E, \mathcal{B}) . The right process X is called *μ -standard*, if it possesses left limits in E \mathbb{P}^μ -a.e. on $(0, \zeta)$ and for every increasing sequence $(T_n)_n$ of stopping times, $T_n \nearrow T$, the sequence $(X_{T_n})_n$ converges to X_T \mathbb{P}^μ -a.e. on $[T < \zeta]$.

Let \mathcal{T} be a topology on E . The right process X is named *càdlàg in the topology* \mathcal{T} \mathbb{P}^μ -a.e. provided that \mathbb{P}^μ -a.e. $t \mapsto X_t$ is right continuous and has left limits in E on $(0, \zeta)$.

For more details on right processes see e.g., [15] and [2].

5.2. Sketch of the proof of Theorem 4.1. Because $\varphi \in C_b(\mathbb{R}^d)$ we may use the Riemann sum approximation of $Y_t(\varphi) : Y_t(\varphi) = \lim_{N \rightarrow \infty} r(N, t)$, where

$$r(knt) := \sum_{k=1}^n \langle \varphi, \widehat{X}_{\frac{k}{N}t} \rangle \frac{t}{N}, \quad N \in \mathbb{N}^*, \quad t \geq 0.$$

We prove first that the following holds

$$(34) \quad \mathbb{E}^\mu \exp(-\langle f, \widehat{X}_t \rangle - Y_t(\varphi)) = \lim_{N \rightarrow \infty} \mathbb{E}^\mu \exp(-r(N-2, t) - \langle (\widetilde{U}_{\frac{t}{N}} \widetilde{W}_{\frac{t}{N}})^2 f, \widehat{X}_{\frac{N-2}{N}t} \rangle).$$

Let us denote $\widetilde{W}_t f := f + t\varphi$, where $\varphi \in b\mathcal{B}(\mathbb{R}^d)$, $t \geq 0$. Using the Markov property we have

$$\begin{aligned} \mathbb{E}^\mu \exp(-\langle f, \widehat{X}_t \rangle - Y_t(\varphi)) &= \lim_{N \rightarrow \infty} \mathbb{E}^\mu \left(\exp(-\langle f + \frac{t}{N}\varphi, \widehat{X}_t \rangle) \cdot \exp(-r(N-1, t)) \right) \\ &= \lim_{N \rightarrow \infty} \mathbb{E}^\mu \left(\exp(-\langle \widetilde{W}_{\frac{t}{N}} f, \widehat{X}_t \rangle) \cdot \exp(-r(N-1, t)) \right) \\ &= \lim_{N \rightarrow \infty} \mathbb{E}^\mu \left(\exp(-r(N-1, t)) \cdot \mathbb{E}^\mu[\exp(-\langle \widetilde{W}_{\frac{t}{N}} f, \widehat{X}_t \rangle) \mid \mathcal{F}_{\frac{N-1}{N}t}] \right) \\ &= \lim_{N \rightarrow \infty} \mathbb{E}^\mu \left(\exp(-r(N-1)) \cdot \mathbb{E}^{\widehat{X}_{\frac{N-1}{N}t}} \exp(-\langle \widetilde{W}_{\frac{t}{N}} f, \widehat{X}_{\frac{t}{N}} \rangle) \right). \end{aligned}$$

We now apply property (26) with $s = \frac{t}{N}$, $\mu = \widehat{X}_{\frac{N-1}{N}}$, and $g = \widetilde{W}_{\frac{t}{N}} f$, to get

$$\begin{aligned} \mathbb{E}^\mu \exp(-\langle f, \widehat{X}_t \rangle - Y_t(\varphi)) &= \lim_{N \rightarrow \infty} \mathbb{E}^\mu \left(\exp(-r(N-1, t)) \exp(-\langle \widetilde{U}_{\frac{t}{N}} \widetilde{W}_{\frac{t}{N}} f, \widehat{X}_{\frac{N-1}{N}t} \rangle) \right) \\ &= \lim_{N \rightarrow \infty} \mathbb{E}^\mu \left(\exp(-r(N-2, t)) \exp(-\langle \frac{t}{N}\varphi + \widetilde{U}_{\frac{t}{N}} \widetilde{W}_{\frac{t}{N}} f, \widehat{X}_{\frac{N-1}{N}t} \rangle) \right) \\ &= \lim_{N \rightarrow \infty} \mathbb{E}^\mu \left(\exp(-r(N-2, t)) \cdot \mathbb{E}^\mu[\exp(-\langle \widetilde{W}_{\frac{t}{N}} \widetilde{U}_{\frac{t}{N}} \widetilde{W}_{\frac{t}{N}} f, \widehat{X}_{\frac{N-1}{N}t} \rangle) \mid \mathcal{F}_{\frac{N-2}{N}t}] \right). \end{aligned}$$

Using again the Markov property and (25) as before we obtain

$$\begin{aligned} \mathbb{E}^\mu \exp(-\langle f, \widehat{X}_t \rangle - Y_t(\varphi)) &= \lim_{N \rightarrow \infty} \mathbb{E}^\mu \left(\exp(-r(N-2, t)) \cdot \mathbb{E}^{\widehat{X}_{\frac{N-2}{N}t}} \exp(-\langle \widetilde{W}_{\frac{t}{N}} C_{\frac{t}{N}} \widetilde{W}_{\frac{t}{N}} f, \widehat{X}_{\frac{t}{N}} \rangle) \right) \\ &= \lim_{N \rightarrow \infty} \mathbb{E}^\mu \left(\exp(-r(N-2, t)) \exp(-\langle (\widetilde{U}_{\frac{t}{N}} \widetilde{W}_{\frac{t}{N}})^2 f, \widehat{X}_{\frac{N-2}{N}t} \rangle) \right), \end{aligned}$$

so, (34) holds.

As in [12], repeating the above procedure we have

$$\mathbb{E}^\mu \exp(-\langle f, \widehat{X}_t \rangle - Y_t(\varphi)) = \lim_{N \rightarrow \infty} \exp(-\langle (\widetilde{U}_{\frac{t}{N}} \widetilde{W}_{\frac{t}{N}})^N f, \mu \rangle).$$

We now use the following Trotter-Lee formula.

Lemma 5.1. *For any nonnegative $\varphi, f \in L^\infty(\mathbb{R}^d)$ the following holds in $L^\infty(\mathbb{R}^d)$:*

$$(35) \quad \widetilde{V}_t(\varphi, f) = \lim_{N \rightarrow \infty} (\widetilde{U}_{t/N} \widetilde{W}_{t/N})^N f, \quad \forall t > 0.$$

This gives us that

$$\lim_{N \rightarrow \infty} \exp(-\langle (\tilde{U}_{\frac{t}{N}} \tilde{W}_{\frac{t}{N}})^N f, \mu \rangle) = \exp(-\langle \tilde{V}_t(\varphi, f), \mu \rangle).$$

The proof of Theorem 4.1 is now finished.

We now prove Lemma 5.1. Let recall that $\tilde{W}_t f := f + t\varphi$, where $\varphi \in b\mathcal{B}(\mathbb{R}^d)$, $t \geq 0$. Clearly $\tilde{W}_t f$ is the solution of the ordinary differential equation

$$\begin{cases} \frac{d\tilde{W}}{dt} = \varphi, \\ \tilde{W}(0) = f. \end{cases}$$

Also $\tilde{V}_t(\varphi, f)$, $t \geq 0$ is the solution of equation (27) and $\tilde{U}_t f$ solution of (25). Observe that if $\varphi \equiv 0$ then $\tilde{V}_t(0, f) = \tilde{U}_t(f)$.

Let us observe that $U_t(e^{-f}) = \exp(-\tilde{U}_t(f))$. Denoting by $V_t(\varphi, g)$ the solution of

$$(36) \quad \begin{cases} v_t = \Delta v - v + \sum_{k \geq 1} q_k v^k - v\varphi, & c \in \mathbb{R}^d, t > 0, \\ v(\cdot, 0) = g, \end{cases}$$

we have $V_t(\varphi, e^{-f}) = \exp(-\tilde{V}_t(\varphi, f))$. We set $W_t(g) = ge^{-t\varphi}$. Thus $W_t(e^{-f}) = \exp(-\tilde{W}_t(f))$.

The proof of Lemma 5.1 is done in two steps. In the first one we prove that is sufficient to show that for any nonnegative $\varphi, \in L^\infty(\mathbb{R}^d)$, we have

$$(37) \quad \|V_t(\varphi, e^{-f}) - (U_{t/n} W_{t/n})^n(e^{-f})\|_{L^\infty(\mathbb{R}^d)} \rightarrow 0, \quad n \rightarrow \infty.$$

The second step consists in proving (37) by checking the conditions in [9, Section 2]. We start with the following estimate for the solutions of equation (36).

Lemma 5.2. *For any nonnegative $\varphi, f \in L^\infty(\mathbb{R}^d)$, the solution of (36) satisfies*

$$e^{-t(1+\|\varphi\|_{L^\infty(\mathbb{R}^d)})} e^{-\|f\|_{L^\infty(\mathbb{R}^d)}} \leq V_t(\varphi, e^{-f}) \leq 1.$$

Proof. We use the maximum principle and the fact v solution of (37) satisfies

$$\Delta v - v(1 + \|\varphi\|_{L^\infty(\mathbb{R}^d)}) \leq v_t \leq \Delta v.$$

□

With this result we can now prove that (37) implies (35). Explicit computations shows the following identity

$$(U_{t/n} W_{t/n})^n(e^{-f}) = \exp(-(\tilde{U}_{t/n} \tilde{W}_{t/n})^n(f)).$$

Using that for any $a, b > 0$ we have $|\log a - \log b| \leq |a - b|/(\min\{a, b\})$ we get

$$(38) \quad \|\tilde{V}_t(\varphi, f) - (\tilde{U}_{t/n} \tilde{W}_{t/n})^n(f)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{\|V_t(\varphi, e^{-f}) - (U_{t/n} W_{t/n})^n(e^{-f})\|_{L^\infty(\mathbb{R}^d)}}{\min_{x \in \mathbb{R}^d} \{\min\{V_t(e^{-f}), (U_{t/n} W_{t/n})^n(e^{-f})\}\}}.$$

Observe that for any nonnegative function g we have

$$W_t(g) \geq e^{-t\|\varphi\|_{L^\infty(\mathbb{R}^d)}} \min_{\mathbb{R}^d} \{g\}.$$

Using Lemma 5.2 with $\varphi \equiv 0$ we also obtain

$$U_t(g) \geq e^{-t} \min_{\mathbb{R}^d} \{g\}.$$

The last two estimates and Lemma 5.2 show that

$$(U_{t/n}W_{t/n})^n(e^{-f}) \geq e^{-t(1+\|\varphi\|_{L^\infty(\mathbb{R}^d)})}e^{-\|f\|_{L^\infty(\mathbb{R}^d)}}.$$

Then the denominator of the right hand side of (38) is bounded from below by some positive constant and it is sufficient to prove (37).

We now prove that (37) holds.

Lemma 5.3. *Let us assume that $g \in L^\infty(\mathbb{R}^d)$ with $0 \leq g \leq 1$ and the sequence $(q_k)_{k \geq 1}$ satisfies $\sum_{k \geq 1} k^2 q_k < \infty$. For any $t > 0$ the following holds*

$$(39) \quad \|V_t(\varphi, g) - (U_{t/n}W_{t/n})^n(g)\|_{L^\infty(\mathbb{R}^d)} \rightarrow 0, \quad n \rightarrow \infty.$$

Proof. We first prove the result for $g \in W^{2,\infty}(\mathbb{R}^d)$ with $0 \leq g \leq 1$ and then by $L^\infty(\mathbb{R}^d)$ -stability of the flows V_t and $(U_{t/n}W_{t/n})^n$ we obtain the desired result.

Step I. Stability of the flow V_t . We prove that for any $g_1, g_2 \in L^\infty(\mathbb{R}^d)$ with $0 \leq g_1, g_2 \leq 1$ we have

$$(40) \quad \|V_t(\varphi, g_1) - V_t(\varphi, g_2)\|_{L^\infty(\mathbb{R}^d)} \leq C(\|\varphi\|_{L^\infty(\mathbb{R}^d)}, \sum_{k \geq 1} k q_k, T) \|g_1 - g_2\|_{L^\infty(\mathbb{R}^d)}, \quad \forall t \in [0, T].$$

Denoting by v_1 and v_2 the corresponding solutions we have $0 \leq v_1, v_2 \leq 1$. Thus $p = v_1 - v_2$ satisfies

$$p_t = \Delta p - p + p \sum_{k \geq 1} q_k (v_1^{k-1} + \dots + v_2^{k-1}) - p\varphi.$$

Using the sub-super solutions methods we find that

$$\|p(t)\|_{L^\infty(\mathbb{R}^d)} \leq e^{tC(\|\varphi\|_{L^\infty(\mathbb{R}^d)}, q)} \|g_1 - g_2\|_{L^\infty(\mathbb{R}^d)}.$$

Step I. Local stability of the flow $(U_{t/n}W_{t/n})^n$ in $W^{2,\infty}(\mathbb{R}^d)$. Let us consider $g \in W^{2,\infty}(\mathbb{R}^d)$. The case of $L^\infty(\mathbb{R}^d)$ -norm easily follows since

$$\|W_t(g)\|_{L^\infty(\mathbb{R}^d)} \leq \|g\|_{L^\infty(\mathbb{R}^d)}, \quad \|U_t(g)\|_{L^\infty(\mathbb{R}^d)} \leq \|g\|_{L^\infty(\mathbb{R}^d)}.$$

Let us now analyze the first derivative. Let us consider u solution of (36) with $\varphi = 0$ and $p = u_{x_k}$. It follows that p satisfies the equation

$$p_t = \Delta p - p - \sum_{k \geq 1} k q_k u^{k-1} p.$$

Choosing $\bar{p} = M e^{\alpha t}$ with $\alpha \geq q - 1$ and $M = \|g_{x_k}\|_{L^\infty(\mathbb{R}^d)}$ we obtain that

$$\|(U_t(g))_{x_k}\|_{L^\infty(\mathbb{R}^d)} \leq e^{\alpha t} \|g_{x_k}\|_{L^\infty(\mathbb{R}^d)}.$$

Using that W_t satisfies

$$\|(W_t(g))_{x_k}\|_{L^\infty(\mathbb{R}^d)} \leq \|g_{x_k}\|_{L^\infty(\mathbb{R}^d)} + t \|\varphi_{x_k}\|_{L^\infty(\mathbb{R}^d)} \|g\|_{L^\infty(\mathbb{R}^d)}$$

it follows that

$$\|(U_{t/n}W_{t/n})(g)\|_{\dot{W}^{1,\infty}(\mathbb{R}^d)} \leq e^{\alpha t/n} (\|g\|_{\dot{W}^{1,\infty}(\mathbb{R}^d)} + \frac{t}{n} \|\varphi\|_{\dot{W}^{1,\infty}(\mathbb{R}^d)} \|g\|_{L^\infty(\mathbb{R}^d)}).$$

Iterating the above argument we obtain that

$$\|(U_{t/n}W_{t/n})^n(g)\|_{\dot{W}^{1,\infty}(\mathbb{R}^d)} \leq e^{\alpha t} (\|g\|_{\dot{W}^{1,\infty}(\mathbb{R}^d)} + t \|\varphi\|_{\dot{W}^{1,\infty}(\mathbb{R}^d)} \|g\|_{L^\infty(\mathbb{R}^d)}).$$

Let us now denote $r = p_{x_i}$. Then r satisfies

$$r_t = \Delta r - r + \sum_{k \geq 1} k q_k u^{k-1} r + \sum_{k \geq 1} k(k-1) q_k u^{k-2} p_{x_i} p_{x_k}.$$

Choosing $\bar{r} = M e^{\beta t}$ and taking into account that

$$|p_{x_i} p_{x_k}| \leq e^{2\alpha t} (\|g\|_{\dot{W}^{1,\infty}(\mathbb{R}^d)} + t \|\varphi\|_{\dot{W}^{1,\infty}(\mathbb{R}^d)} \|g\|_{L^\infty(\mathbb{R}^d)})^2 = e^{2\alpha t C}$$

we can choose

$$\beta(g) = q - 1 + \left(\sum_{k \geq 1} k^2 q_k \right) (\|g\|_{\dot{W}^{1,\infty}(\mathbb{R}^d)} + t \|\varphi\|_{\dot{W}^{1,\infty}(\mathbb{R}^d)})^2$$

and

$$M = \|q_{x_k x_i}\|_{L^\infty(\mathbb{R}^d)}$$

to obtain that

$$\|U_t(g)\|_{\dot{W}^{2,\infty}(\mathbb{R}^d)} \leq \|g\|_{\dot{W}^{2,\infty}(\mathbb{R}^d)} \exp(t\beta(g)).$$

Using the stability in $\dot{W}^{1,\infty}(\mathbb{R}^d)$ and iterating the above estimate on U_t we also obtain the local-stability in $\dot{W}^{2,\infty}(\mathbb{R}^d)$.

Consistency. We now prove that for any $g \in W^{2,\infty}$, $0 \leq g \leq 1$ we have

$$\lim_{\epsilon \rightarrow 0} \frac{U_\epsilon W_\epsilon g - g}{\epsilon} - \left(\Delta g - g + \sum_{k \geq 1} q_k u^k - g\varphi \right) = 0.$$

We use that $g \in W^{2,\infty}$, $0 \leq g \leq 1$ implies that our solution satisfies $U \in C^1([0, \infty], L^\infty(\mathbb{R}^d)) \cap C([0, T], W^{2,\infty}(\mathbb{R}^d))$. The explicit form of $W_\epsilon(g) = g e^{-\epsilon\varphi}$ give us the desired result. \square

REFERENCES

- [1] L. Beznea, Potential theoretical methods in the construction of measure-valued branching processes, *J. Europ. Math. Soc.* **13** (2011), 685–707.
- [2] L. Beznea, N. Boboc, *Potential Theory and Right Processes*, Springer Series, Mathematics and Its Applications **572**, Kluwer, Dordrecht, 2004.
- [3] L. Beznea, M. Deaconu, and O. Lupaşcu, Branching processes for the fragmentation equation, *Stochastic Processes and their Applications* **125** (2015), 1861–1885.
- [4] L. Beznea, M. Deaconu, and O. Lupaşcu, Stochastic equation of fragmentation and branching processes related to avalanches, *J. Stat. Physics* **162** (2016), 824–841.
- [5] L. Beznea, O. Lupaşcu, Measure-valued discrete branching Markov processes, *Trans. Amer. Math. Soc.* **368** (2016), 5153–5176.
- [6] L. Beznea, O. Lupaşcu, and A.-G. Oprina, A unifying construction for measure-valued continuous and discrete branching processes. In: *Complex Analysis and Potential Theory*, CRM Proceedings and Lecture Notes, vol. **55**, Amer. Math. Soc., Providence, RI, 2012, pp. 47–59.
- [7] L. Beznea, A.-G. Oprina, Nonlinear PDEs and measure-valued branching type processes, *J. Math. Anal. Appl.* **384** (2011), 16–32.
- [8] L. Beznea, A.-G. Oprina, Bounded and L^p -weak solutions for nonlinear equations of measure-valued branching processes, *Nonlinear Analysis* **107** (2014), 34–46.
- [9] A.J. Chorin, T.J.R. Hughes, M.F. McCracken, and J.E. Marsden, Product formulas and numerical algorithms, *Communications on Pure and Applied Mathematics* **31** (1978), 205–256.
- [10] M. Escobedo, E. Zuazua, Large time behavior for convection-diffusion equations in \mathbb{R}^N . *Journal of Functional Analysis* **100** (1991), 119–161.
- [11] N. Ikeda, M. Nagasawa, and S. Watanabe, Branching Markov processes I, *J. Math. Kyoto Univ.* **8** (1968), 233–278.
- [12] I. Iscoe, A weighted occupation time for a class of measure-valued branching processes. *Probab. Th. Rel. Fields* **71** (1986), 85–116.

- [13] Z. H. Li, *Measure-Valued Branching Markov Processes*. Probab. Appl., Springer, 2011.
- [14] O. Lupaşcu, V. Stănculescu, Numerical solution for the non-linear Dirichlet problem of a branching process, *Complex Analysis and Op. Th.* (2017), DOI: 10.1007/s11785-017-0642-z, to appear.
- [15] M. Sharpe, *General Theory of Markov Processes*. Academic Press, Boston, 1988.

L. BEZNEA

SIMION STOILOW INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY,
P.O. BOX 1-764, RO-014700, BUCHAREST, ROMANIA, AND
UNIVERSITY OF BUCHAREST, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
E-mail address: `lucian.beznea@imar.ro`

L. I. IGNAT

SIMION STOILOW INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY,
P.O. BOX 1-764, RO-014700, BUCHAREST, ROMANIA, AND
UNIVERSITY OF BUCHAREST, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
E-mail address: `liviu.ignat@gmail.com`
Web page: `http://www.imar.ro/~lignat`

J. D. ROSSI

DPTO. DE MATEMÁTICAS, FCEYN, UNIVERSIDAD DE BUENOS AIRES,
1428, BUENOS AIRES, ARGENTINA.
E-mail address: `jrossi@dm.uba.ar`
Web page: `http://mate.dm.uba.ar/~jrossi/`